Calculate the limit
https://www.linkedin.com/groups/8313943/8313943-6369422736149794819

$$\lim_{n\to\infty} \frac{\sqrt[n]{\sqrt{2!}\cdot\sqrt[3]{3!}\cdot\ldots\cdot\sqrt[n]{n!}}}{\sqrt[n+1]{(2n+1)!!}}.$$

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First note that $\displaystyle\lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}} = e$ and $\displaystyle\lim_{n\to\infty}\frac{n}{\sqrt[n]{(2n-1)!!}} = \frac{e}{2}$.

Indeed, let $a_n := \dfrac{n^n}{n!}$ and $b_n := \dfrac{n^n}{(2n-1)!!}, n \in \mathbb{N}$.

Since $\dfrac{n}{\sqrt[n]{n!}} = \sqrt[n]{a_n}$, $\dfrac{n}{\sqrt[n]{(2n-1)!!}} = \sqrt[n]{b_n}$ and

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{-n} = e,$$

$$\lim_{n\to\infty}\frac{b_{n+1}}{b_n} = \lim_{n\to\infty}\left(\frac{n+1}{2n+1}\cdot\left(1+\frac{1}{n}\right)^{n}\right) = \frac{e}{2}, \text{ then}$$

by Multiplicative Stolz-Cezaro Theorem $\displaystyle\lim_{n\to\infty}\sqrt[n]{a_n} = e$ and $\displaystyle\lim_{n\to\infty}\sqrt[n]{b_n} = \frac{e}{2}$.

Using that we obtain

$$\lim_{n\to\infty}\frac{\sqrt[n]{\sqrt{2!}\cdot\sqrt[3]{3!}\cdot\ldots\cdot\sqrt[n]{n!}}}{\sqrt[n+1]{(2n+1)!!}} = \lim_{n\to\infty}\left(\frac{n+1}{\sqrt[n+1]{(2n+1)!!}}\cdot\frac{\sqrt[n]{\sqrt{2!}\cdot\sqrt[3]{3!}\cdot\ldots\cdot\sqrt[n]{n!}}}{n}\cdot\frac{n}{n+1}\right) =$$

$\displaystyle\frac{e}{2}\lim_{n\to\infty}\sqrt[n]{c_n}$, where $c_n := \dfrac{\sqrt{2!}\cdot\sqrt[3]{3!}\cdot\ldots\cdot\sqrt[n]{n!}}{n^n}$ and, since

$$\lim_{n\to\infty}\frac{c_{n+1}}{c_n} = \lim_{n\to\infty}\left(\frac{\sqrt[n+1]{(n+1)!}}{n+1}\cdot\left(1+\frac{1}{n}\right)^{-n}\right) = e^{-2},$$

then by Multiplicative Stolz-Cezaro Theorem $\displaystyle\lim_{n\to\infty}\sqrt[n]{c_n} = e^{-2}$.

Thus, $\displaystyle\lim_{n\to\infty}\frac{\sqrt[n]{\sqrt{2!}\cdot\sqrt[3]{3!}\cdot\ldots\cdot\sqrt[n]{n!}}}{\sqrt[n+1]{(2n+1)!!}} = \frac{e}{2}\cdot\frac{1}{e^2} = \frac{1}{2e}$.